

A proof of the Kazdan-Warner identity via the Minkowski spacetime

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Abstract

Any 2-dim Riemannian manifold with spherical topology can be embedded isometrically into a lightcone of the Minkowski spacetime. We apply this fact to give a proof of the Kazdan-Warner identity.

1 The Kazdan-Warner identity

$(\mathbb{S}^2, g_{\mathbb{S}^2})$ is the sphere of radius 1 at the origin in the 3-dim Euclidean space \mathbb{E}^3 . $(\mathbb{S}^2, g_{\mathbb{S}^2})$ has the constant Gauss curvature 1. $\{x_1, x_2, x_3\}$ is the rectangular coordinate system of \mathbb{E}^3 . We denote ∇ the Levi-Civita connection on $(\mathbb{S}^2, g_{\mathbb{S}^2})$. Then the vector fields $\nabla x_1, \nabla x_2, \nabla x_3$ are conformal Killing vector fields on $(\mathbb{S}^2, g_{\mathbb{S}^2})$, i.e. the diffeomorphisms generated by them are conformal.

g is another Riemannian metric on the sphere \mathbb{S}^2 . By the uniformization theorem, there exists a function f on \mathbb{S}^2 such that the conformal metric $e^{-2f}g$ has the constant Gauss curvature 1. Hence there is a diffeomorphism $\psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $\psi^*g = e^{2f \circ \psi}g_{\mathbb{S}^2}$.

In the following of this note, we assume that g is conformal to the standard metric $g_{\mathbb{S}^2}$ with the conformal factor e^{2f} . Let K_g be the Gauss curvature of (\mathbb{S}^2, g) . Then we have the following from [2]:

The Kazdan-Warner identity.

$$\int_{\mathbb{S}^2} \langle \nabla K_g, \nabla x_i \rangle_{g_{\mathbb{S}^2}} d\text{vol}_g = 0. \quad (1.1)$$

We denote $\tilde{\nabla}$ the Levi-Civita connection on (\mathbb{S}^2, g) . Then we can rewrite the Kazdan-Warner identity as following

$$\int_{\mathbb{S}^2} \langle \tilde{\nabla} K_g, \nabla x_i \rangle_g d\text{vol}_g = 0. \quad (1.2)$$

Since ∇x_i is a conformal Killing vector field for $g_{\mathbb{S}^2}$, it is also a conformal Killing vector field for g . Actually we have that for any conformal Killing vector field X on (\mathbb{S}^2, g) ,

$$\int_{\mathbb{S}^2} \langle \tilde{\nabla} K_g, X \rangle_g d\text{vol}_g = 0. \quad (1.3)$$

We will give a proof of (1.3), thus the Kazdan-Warner identity follows.

2 The Minkowski spacetime

(\mathbb{M}, η) is the 4-dim Minkowski spacetime. $\{x_0, x_1, x_2, x_3\}$ is the rectangular coordinate system of \mathbb{M} . The metric $\eta = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$.

t, r are two functions on \mathbb{M} : $t = x_0$ and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. We define the optical functions u, v by $u = \frac{1}{2}(t - r)$ and $v = \frac{1}{2}(t + r)$. So $t = u + v$ and $r = v - u$.

Let C_u be the level set of u , which is the future lightcone at the point $(u, 0, 0, 0)$. Let \underline{C}_v be the level set of v , which is the past lightcone at the point $(v, 0, 0, 0)$. Let $S_{u,v}$ be the intersection of C_u and \underline{C}_v . $S_{u,v}$ is a sphere of radius $r = v - u$.

We define a map $\phi : \mathbb{M} \setminus \{r = 0\} \rightarrow \mathbb{S}^2$ by $\phi : x = (x_0, x_1, x_2, x_3) \mapsto (\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r})$. $\mathbb{R}_{v>u}^2$ is the open half-plane $\{(v, u) \in \mathbb{R}^2 \mid v > u\}$. Then we have another coordinate system Φ on $\mathbb{M} \setminus \{r = 0\}$ given by

$$\begin{aligned} \Phi : \quad \mathbb{M} \setminus \{r = 0\} &\rightarrow \mathbb{R}_{v>u}^2 \times \mathbb{S}^2, \\ x = (x_0, x_1, x_2, x_3) &\mapsto \left((v, u), \phi(x) = \left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r} \right) \right). \end{aligned} \quad (2.1)$$

Let ∂_t be the vector field ∂_0 and ∂_r be the vector field $\frac{x_1}{r}\partial_1 + \frac{x_2}{r}\partial_2 + \frac{x_3}{r}\partial_3$. In the coordinate system Φ , the coordinate vector fields $\partial_v = \partial_t + \partial_r$ and $\partial_u = \partial_t - \partial_r$.

∂_u and ∂_v are both null vector fields, i.e. $\eta(\partial_u, \partial_u) = \eta(\partial_v, \partial_v) = 0$. The inner product of ∂_u and ∂_v is -2 , i.e. $\eta(\partial_u, \partial_v) = -2$. We see that ∂_u and ∂_v are orthonormal to the tangent space of any $S_{u,v}$. So in the coordinate system Φ , the metric $\eta = -2(du \otimes dv + dv \otimes du) + r^2 g_{\mathbb{S}^2}$.

In particular, when we restrict the coordinate system Φ on the lightcone $C_0 \setminus \{o\}$, we get a coordinate system of $C_0 \setminus \{o\}$:

$$\Phi|_{C_0 \setminus \{o\}} : C_0 \setminus \{o\} \rightarrow \mathbb{R}_+ \times \mathbb{S}^2, \quad x \in C_0 \setminus \{o\} \mapsto (v, \phi(x)). \quad (2.2)$$

The induced metric $\eta|_{C_0 \setminus \{o\}} = v^2 g_{\mathbb{S}^2}$ is degenerated.

3 The isometric embedding of (\mathbb{S}^2, g) into a lightcone of the Minkowski spacetime

Via the coordinate system $\Phi|_{C_0 \setminus \{o\}}$, we can represent any closed spacelike surface in $C_0 \setminus \{o\}$ as a graph of a function on \mathbb{S}^2 . S is a closed spacelike surface in $C_0 \setminus \{o\}$, there exists a unique function h on \mathbb{S}^2 such that

$$\Phi|_{C_0 \setminus \{o\}}(S) = \left\{ (e^{h(\theta)}, \theta) \in \mathbb{R}_+ \times \mathbb{S}^2 \mid \theta \in \mathbb{S}^2 \right\}. \quad (3.1)$$

Conversely for any function h on \mathbb{S}^2 , the map

$$\psi_h : \mathbb{S}^2 \rightarrow \mathbb{R}^+ \times \mathbb{S}^2, \quad \theta \in \mathbb{S}^2 \mapsto (e^{h(\theta)}, \theta) \quad (3.2)$$

is an embedding of \mathbb{S}^2 into $C_0 \setminus \{o\}$. Its image

$$S_h = \psi_h(\mathbb{S}^2) = \left\{ (e^{h(\theta)}, \theta) \in \mathbb{R}_+ \times \mathbb{S}^2 \mid \theta \in \mathbb{S}^2 \right\} \quad (3.3)$$

is a closed spacelike surface in $C_0 \setminus \{o\}$. Moreover, $\psi_h^*(\eta|_{S_h}) = e^{2h} g_{\mathbb{S}^2}$ since $\eta|_{C_0 \setminus \{o\}} = v^2 g_{\mathbb{S}^2}$.

Hence, we can embed $(\mathbb{S}^2, g = e^{2f} g_{\mathbb{S}^2})$ isometrically into $C_0 \setminus \{o\}$ as above by taking $h = f$.

4 The geometry of a spacelike surface in the Minkowski spacetime

Let S be a orientable spacelike surface in the Minkowski spacetime (\mathbb{M}, η) . TS is the tangent bundle and NS is the normal bundle of S in (\mathbb{M}, η) .

Since S is spacelike, $\eta|_{TS}$ is positive definite and $\eta|_{NS}$ is of signature $(1, 1)$. Hence we can choose a null frame $\{L, \underline{L}\}$ of NS such that L and \underline{L} are both future-directed null vector fields and their inner product is -2 , i.e.

$$\eta(L, L) = \eta(\underline{L}, \underline{L}) = 0, \quad \eta(L, \underline{L}) = -2. \quad (4.1)$$

Such a choice isn't unique, since for any positive function a on S , the frame $\{aL, a^{-1}\underline{L}\}$ also satisfies the above conditions.

In the following, we fix such a null frame $\{L, \underline{L}\}$. Then we can choose an orthonormal frame $\{e_1, e_2\}$ of TS at least locally such that $\{e_1, e_2, e_3 = \underline{L}, e_4 = L\}$ is positive oriented in \mathbb{M} . Choose the orientation of S to be the orientation of $\{e_1, e_2\}$. Let $A, B = 1, 2$.

The intrinsic geometry of S is given by the induced metric $\eta|_S$. Let ∇ be the Levi-Civita connection on $(S, \eta|_S)$ and d be the exterior derivative on S . Let ϵ be the volume form on $(S, \eta|_S)$. We define the intrinsic differential operators curl , div on S by

$$\text{curl} \omega = \epsilon^{AB} \nabla_A \omega_B, \quad \text{for any 1-form } \omega \text{ on } S; \quad (4.2)$$

$$(\text{div} T)_A = \nabla^B T_{BA}, \quad \text{for any symmetric 2-tensor field } T \text{ on } S. \quad (4.3)$$

The extrinsic geometry of S in (\mathbb{M}, η) is given by the null second fundamental forms $\chi, \underline{\chi}$ and the torsion ζ defined as following:

$$\chi(X, Y) = \eta(\nabla_X L, Y), \quad \text{for any } X, Y \in TS; \quad (4.4)$$

$$\underline{\chi}(X, Y) = \eta(\nabla_X \underline{L}, Y), \quad \text{for any } X, Y \in TS; \quad (4.5)$$

$$\zeta(X) = \frac{1}{2} \eta(\nabla_X L, \underline{L}), \quad \text{for any } X \in TS. \quad (4.6)$$

$\chi, \underline{\chi}$ are covariant symmetric 2-tensor fields and ζ is a 1-form on S . Let $\text{tr}\chi$ and $\text{tr}\underline{\chi}$ be the traces of χ and $\underline{\chi}$ on S , i.e. the contractions with $\eta|_S^{-1}$ on TS . Let $\hat{\chi}$ and $\hat{\underline{\chi}}$ be the tracefree parts of χ and $\underline{\chi}$ on S .

In analogy with the Gauss equation and Codazzi equation for a surface in 3-dim Euclidean spacetime, we have the following equations:

The Gauss equations

$$K + \frac{1}{4}\text{tr}\chi \text{tr}\underline{\chi} - \frac{1}{2}(\hat{\chi}, \hat{\underline{\chi}})_{\eta|S} = 0, \quad (4.7)$$

$$\text{curl}\zeta + \frac{1}{2}\hat{\underline{\chi}} \wedge \hat{\chi} = 0; \quad (4.8)$$

The Codazzi equations

$$\text{div}\hat{\chi} - \frac{1}{2}\not{d}\text{tr}\chi + \hat{\chi} \cdot \zeta - \frac{1}{2}\text{tr}\chi \zeta = 0, \quad (4.9)$$

$$\text{div}\hat{\underline{\chi}} - \frac{1}{2}\not{d}\text{tr}\underline{\chi} - \hat{\underline{\chi}} \cdot \zeta + \frac{1}{2}\text{tr}\underline{\chi} \zeta = 0; \quad (4.10)$$

where

$$(\hat{\chi}, \hat{\underline{\chi}})_{\eta|S} = (\eta|_S^{-1})^{AC}(\eta|_S^{-1})^{BD}\hat{\chi}_{AB}\hat{\underline{\chi}}_{CD}, \quad \hat{\underline{\chi}} \wedge \hat{\chi} = \epsilon^{AB}\hat{\underline{\chi}}_A{}^C\hat{\chi}_{CB}, \quad (4.11)$$

and

$$(\hat{\chi} \cdot \zeta)_A = \hat{\chi}_A{}^B\zeta_B, \quad (\hat{\underline{\chi}} \cdot \zeta)_A = \hat{\underline{\chi}}_A{}^B\zeta_B. \quad (4.12)$$

One can find the proofs of these equations in [1].

We can apply these equations to S in the lightcone $C_0 \setminus \{o\}$. We see ∂_v along S is a null vector field in NS . So we choose L to be the null vector field ∂_v over S . Then we can find \underline{L} in NS such that $\{L, \underline{L}\}$ is a null frame of NS .

Since the metric $\eta|_{C_0 \setminus \{o\}} = v^2 g_{\mathbb{S}^2}$, we see that the induced metric on S deforms conformally when we deform S in the direction of ∂_v . This means that the seconded fundamental form χ of S is a multiple of the induced metric $\eta|_S$. Hence $\hat{\chi} = 0$. So on S , the equations are simpler:

$$K + \frac{1}{4}\text{tr}\chi \text{tr}\underline{\chi} = 0, \quad (4.13)$$

$$\text{curl}\zeta = 0, \quad (4.14)$$

$$\not{d}\text{tr}\chi + \text{tr}\chi \zeta = 0, \quad (4.15)$$

$$\text{div}\hat{\underline{\chi}} - \frac{1}{2}\not{d}\text{tr}\underline{\chi} - \hat{\underline{\chi}} \cdot \zeta + \frac{1}{2}\text{tr}\underline{\chi} \zeta = 0. \quad (4.16)$$

5 The proof of the Kazdan-Warner identity

S is a closed surface in $C_0 \setminus \{o\}$. Let X be a conformal Killing vector field on $(S, \eta|_S)$. The deformation tensor field ${}^{(X)}\pi = \mathcal{L}_X \eta|_S$ of X is a multiple of $\eta|_S$. Then ${}^{(X)}\pi = {}^{(X)}\Omega \cdot \eta|_S$ where ${}^{(X)}\Omega = \frac{1}{2} \text{div} X$. We define the operator sym by

$$(\text{sym } T)_{AB} = T_{AB} + T_{BA}, \text{ for any covariant 2-tensor field } T \text{ on } S. \quad (5.1)$$

Then $\text{sym}(\nabla X) = \mathcal{L}_X \eta|_S = {}^{(X)}\pi = \Omega^X \cdot \eta|_S$.

$$\begin{aligned} \int_S \langle \nabla K, X \rangle_{\eta|_S} \text{dvol}_{\eta|_S} &\stackrel{(4.13)}{=} \int_S -\frac{1}{4} \langle \nabla(\text{tr} \chi \text{tr} \underline{\chi}), X \rangle_{\eta|_S} \text{dvol}_{\eta|_S} \\ &= \int_S -\frac{1}{4} \langle \text{tr} \underline{\chi} \nabla \text{tr} \chi + \text{tr} \chi \nabla \text{tr} \underline{\chi}, X \rangle_{\eta|_S} \text{dvol}_{\eta|_S} \\ &\stackrel{(4.15)(4.16)}{=} \int_S -\frac{1}{2} \{ \text{tr} \chi \text{div} \hat{\chi} \cdot X - \hat{\chi}(\text{tr} \chi \zeta, X) \} \text{dvol}_{\eta|_S} \\ &= \int_S \frac{1}{2} \{ \text{tr} \chi \langle \hat{\chi}, \nabla X \rangle_{\eta|_S} + \hat{\chi}(\nabla \text{tr} \chi + \text{tr} \chi \zeta, X) \} \text{dvol}_{\eta|_S} \\ &\stackrel{(4.15)}{=} \int_S \frac{1}{4} \text{tr} \chi \langle \hat{\chi}, \text{sym}(\nabla X) \rangle_{\eta|_S} \text{dvol}_{\eta|_S} \\ &= \int_S \frac{1}{4} \text{tr} \chi \langle \hat{\chi}, \Omega^X \cdot \eta|_S \rangle_{\eta|_S} \text{dvol}_{\eta|_S} \\ &= 0. \end{aligned}$$

The last equality follows from that $\hat{\chi}$ is tracefree.

Together with the constructions in section 3, we prove (1.3).

6 The gauge transformations on the normal bundle of a spacelike surface in the Minkowski spacetime

Recall that in the section 4, we introduced the normal bundle NS of a oriented spacelike surface S in the Minkowski spacetime (\mathbb{M}, η) . Since that for any $p \in S$, the normal space $N_p S$ endowed with the induced metric $\eta|_{N_p S}$ is isometric to the 2-dim Minkowski spacetime, we have a 1-dim non-compact abelian group of isometries for $(N_p S, \eta|_{N_p S})$. The group of isometries on $(N_p S, \eta|_{N_p S})$ is just the group of Lorentz rotations of the 2-dim Minkowski spacetime. We can explicitly write down the isometries via the null frame $\{L, \underline{L}\}$. Any positive number $a \in \mathbb{R}_{>0}$, we have the mapping $\mathcal{L}_a : N_p S \rightarrow N_p S$ defined by

$$\mathcal{L}_a : \quad L_p \rightarrow aL_p, \quad \underline{L}_p \rightarrow a^{-1}\underline{L}_p. \quad (6.1)$$

The group structure is simply given by $\mathcal{L}_a \circ \mathcal{L}_b = \mathcal{L}_{ab}$ for any $a, b \in \mathbb{R}_{>0}$.

So the normal bundle $(NS, \eta|_{NS})$ is a vector bundle on S with the group action of $(\mathbb{R}_{>0}, \cdot)$. Moreover, the null frame bundle of $(NS, \eta|_{NS})$ is a principal $(\mathbb{R}_{>0}, \cdot)$ -bundle. This principal bundle is actually trivial since we can find a global section, which is a global null frame. The parallel transport on the normal bundle NS defines a principal connection on the null frame bundle. We see that the torsion ζ of a null frame $\{L, \underline{L}\}$ is actually the connection 1-form for this principal connection. Assume now that a is a positive function over S , then $\{aL, a^{-1}\underline{L}\}$ is another null frame of $(NS, \eta|_{NS})$. Let us denote ζ_a being the torsion for $\{aL, a^{-1}\underline{L}\}$. Direct calculation by the definition of the torsion (4.6) shows that

$$\zeta_a = \zeta - a^{-1}d a = \zeta - d \log a, \quad (6.2)$$

which is just the transformation formula for the connection form. However $\text{curl} \zeta_a = \text{curl} \zeta$ keeps invariant, because it is actually the curvature of this connection. In particular, we can choose a positive function a such that

$$d\text{iv} \zeta_a = d\text{iv} \zeta - d \log a = 0. \quad (6.3)$$

Now Assume that the oriented spacelike surface S is contained in the lightcone $C_0 \setminus \{o\}$. We take the null frame $\{L, \underline{L}\}$ of $(NS_h, \eta|_{NS_h})$ such that its torsion ζ satisfies $d\text{iv} \zeta = 0$. Associated with this null frame $\{L, \underline{L}\}$, we have the Gauss equations and Codazzi equations. Since $\hat{\chi} = 0$ still holds, we have $\text{curl} \zeta = 0$. Then the equations

$$d\text{iv} \zeta = 0, \quad \text{curl} \zeta = 0, \quad (6.4)$$

imply that $\zeta = 0$. Then we have

$$K + \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} = 0, \quad (6.5)$$

$$d \text{tr} \chi = 0, \quad (6.6)$$

$$d\text{iv} \hat{\chi} - \frac{1}{2} d \text{tr} \underline{\chi} = 0. \quad (6.7)$$

Hence $\text{tr} \chi$ is a constant function over S and we can assume $\text{tr} \chi \equiv 1$ since we can always achieve this by modifying the null frame by a positive constant. So we get that

$$dK = -\frac{1}{2} d\text{iv} \hat{\chi}. \quad (6.8)$$

We consider the following operator $d\text{iv}$ taking a 2-covariant symmetric, traceless tensor ξ into the 1-form $d\text{iv} \xi$. The L^2 -adjoint of $d\text{iv}$ is the operator taking a 1-form f into the 2-covariant symmetric, traceless tensor $-\frac{1}{2} \widehat{\mathcal{L}_{f^\#} \eta}|_S$, where $\widehat{\mathcal{L}_{f^\#} \eta}|_S$ is the traceless part of the

Lie derivative of $\eta|_S$ with respect to the vector field f^\sharp . This can be shown as the following:

$$\int_S \langle \text{div} \xi, f \rangle \text{dvol}_{\eta|_S} = \int_S \langle \xi, -\nabla f \rangle \text{dvol}_{\eta|_S} \quad (6.9)$$

$$= \int_S \langle \xi, -\frac{1}{2} \text{sym} \nabla f \rangle \text{dvol}_{\eta|_S} \quad (6.10)$$

$$= \int_S \langle \xi, -\frac{1}{2} \mathcal{L}_{f^\sharp} \eta|_S \rangle \text{dvol}_{\eta_S} \quad (6.11)$$

$$= \int_S \langle \xi, -\frac{1}{2} \widehat{\mathcal{L}_{f^\sharp} \eta|_S} \rangle \text{dvol}_{\eta_S}. \quad (6.12)$$

The kernel of the L^2 -adjoint of div consists of the 1-form f such that f^\sharp is a conformal Killing vector field. Since the range of div is L^2 orthogonal to the kernel of its L^2 -adjoint, then the identity (1.3) follows from (6.8).

References

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